Monodromy Groups of Belyĭ Lattès Maps

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A rational map $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ from the Riemann Sphere to itself is said to be a Lattès Map if there are "well-behaved" maps $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\psi : E(\mathbb{C}) \to E(\mathbb{C})$ such that $\gamma \circ \phi = \phi \circ \psi$. We are interested in those Lattès Maps which are also Belyĭ Maps and their associated monodromy groups.

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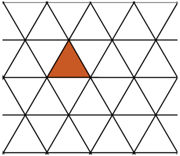
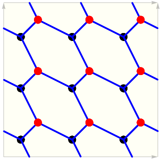
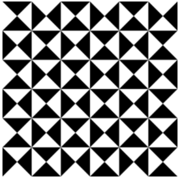
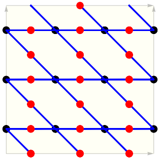
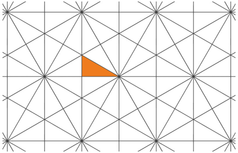
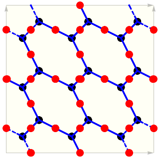
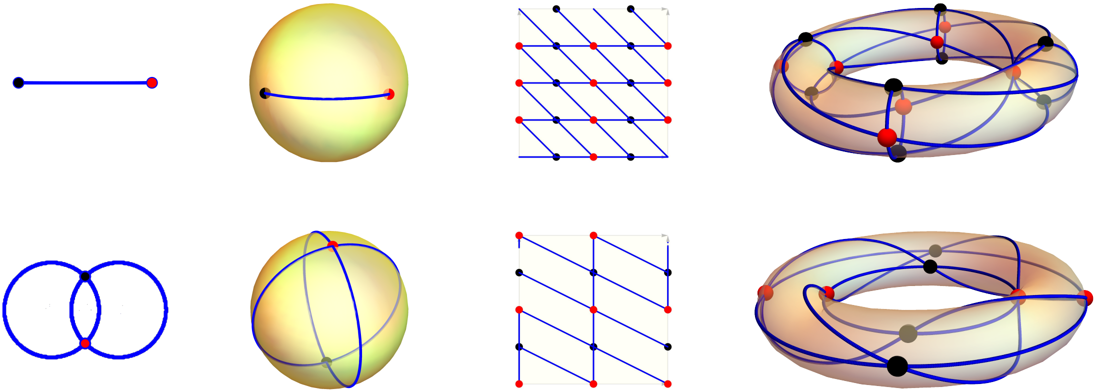
Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors, and do not necessarily reflect the views of the National Science Foundation.

Introduction and Motivation: Regular Tilings

Triangles	Tilings of the Euclidean Plane	Tilings of the Torus
$60^\circ - 60^\circ - 60^\circ$		
$45^\circ - 45^\circ - 90^\circ$		
$30^\circ - 60^\circ - 90^\circ$		

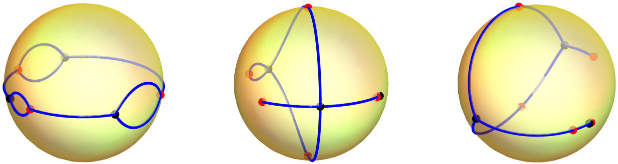
A **Dessin d'Enfants** is a connected graph uniquely determined by a rational function ϕ. On this graph, each black vertex represents the preimage $\phi^{-1}(\{0\})$, each red vertex represents the preimage $\phi^{-1}(\{1\})$, and each edge is the preimage $\phi^{-1}(\{(0,1)\})$.



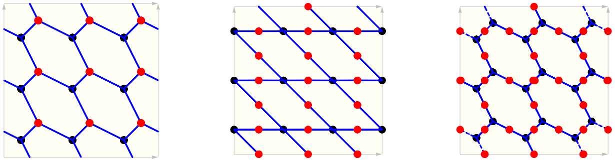
Main Question

How are the Dessins on the sphere and torus related?

Rational map $\gamma(z)$

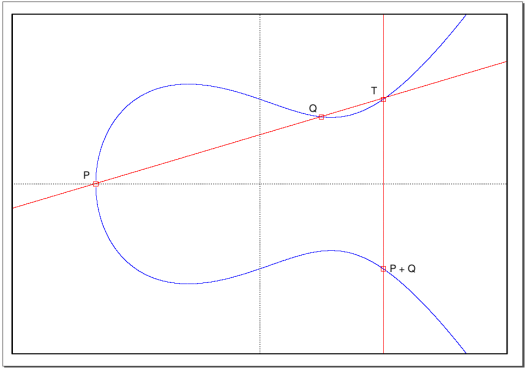


Rational map $\beta(x, y) = \gamma \circ \phi(x, y)$



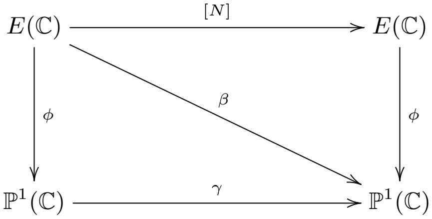
Definitions

- An **elliptic curve** is a non-singular curve of genus 1 of the form $y^2 = x^3 + A\,x + B$ where $A, B \in \mathbb{C}$.
- Let $S = E(\mathbb{C})$ be the collection of complex numbers x_0 and y_0 satisfying $y^2 = x^3 + A\,x + B$ along with the "point at infinity" O_E. This is a torus. In particular, it is a compact, connected Riemann surface.
- Let P, Q, and $P * Q$ be points on E which lie on a line. Then the binary operation $P \oplus Q = \big(P * Q\big) * O_E$ turns $\big(E(\mathbb{C}), \oplus\big)$ into an abelian group.



Definitions

- The **Riemann sphere** $\mathbb{P}^1(\mathbb{C})$ is defined as $\mathbb{C} \cup \{\infty\}$.
- A **Belyĭ map** $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic (rational) function defined on a compact, connected Riemann surface S which is ramified over at most three points. We choose these points to be 0, 1, and ∞.
- A **Lattès map** $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function satisfying $\gamma \circ \phi = \phi \circ [N]$ for some meromorphic function $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and "multiplication-by-N" isogeny $[N] : E(\mathbb{C}) \to E(\mathbb{C})$ where $[N]P = P \oplus \cdots \oplus P$.



Theorem (Ayberk Zeytin, 2021; PRiME 2022)

Assume $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Lattès map.

- **If ϕ is a Belyĭ map, then both γ and $\beta = \gamma \circ \phi = \phi \circ [N]$ are Belyĭ maps as well.**
- Any Belyĭ Lattès map arises from one of three families:

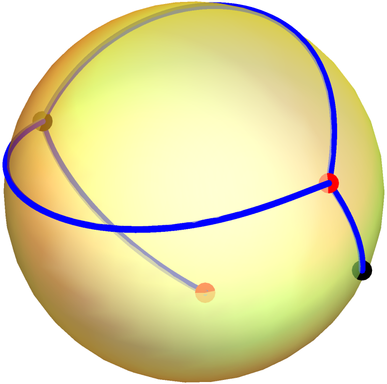
Elliptic Curve	Belyĭ Map ϕ	Degree of ϕ
$E : y^2 = x^3 + B$	$\phi(x,y) = \dfrac{\sqrt{B} - y}{2\sqrt{B}}$	$\deg(\phi) = 3$
$E : y^2 = x^3 + A\,x$	$\phi(x,y) = -\dfrac{x^2}{A}$	$\deg(\phi) = 4$
$E : y^2 = x^3 + B$	$\phi(x,y) = -\dfrac{x^3}{B}$	$\deg(\phi) = 6$

Definition

Fix a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ for a compact, connected Riemann surface S.

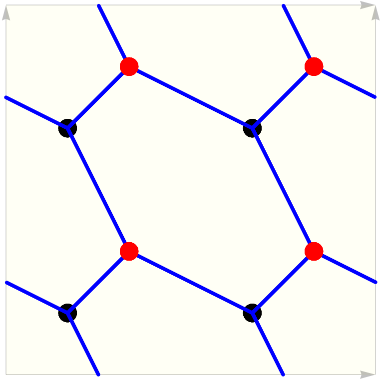
- Since β is ramified at 0, 1, and ∞, then define the **branch points** as the preimages $B = \beta^{-1}\big(\{0\}\big)$, $R = \beta^{-1}\big(\{1\}\big)$, and $F = \beta^{-1}\big(\{\infty\}\big)$.
- Let the **ramification index** e_P of a branch point $P \in B \cup R \cup F$ be the number of edges that stem from the point.
- The **degree sequence** is the multiset $\mathcal{D} = \Big\{ \{e_P \,\big|\, P \in B\},\ \{e_P \,\big|\, P \in R\},\ \{e_P \,\big|\, P \in F\} \Big\}$.

$$\gamma(z) = \frac{(z-1)\,(z+1)^3}{(2\,z-1)^3}$$



$$\mathcal{D} = \Big\{ \{1,3\},\ \{1,3\},\ \{1,3\} \Big\}$$

$$\beta(x,y) = \gamma \circ \phi(x,y) = \frac{(1+y)\,(3-y)^3}{16y^3}$$



$$\mathcal{D} = \Big\{ \{3,3,3,3\},\ \{3,3,3,3\},\ \{3,3,3,3\} \Big\}$$

Degree Sequence of $\beta = \gamma \circ \phi = \phi \circ [N]$

- Let the composition map β be determined by Belyĭ map ϕ and multiplication-by-N map $[N]$.

$\deg \phi$	e_p for $\beta^{-1}(0)$	$\lvert\beta^{-1}(0)\rvert$	e_p for $\beta^{-1}(1)$	$\lvert\beta^{-1}(1)\rvert$	e_p for $\beta^{-1}(\infty)$	$\lvert\beta^{-1}(\infty)\rvert$
3	3	N^2	3	N^2	3	N^2
4	4	N^2	2	$2N^2$	4	N^2
6	3	$2N^2$	2	$3N^2$	6	N^2

Degree Sequence of β for $\deg \phi = 6$

$$\mathcal{D} = \{\underbrace{\{3, \ldots, 3\}}_{2N^2 \text{ copies}}, \underbrace{\{2, \ldots, 2\}}_{3N^2 \text{ copies}}, \underbrace{\{6, \ldots, 6\}}_{N^2 \text{ copies}}\}.$$

Degree Sequence of Composition Map β for $\deg\phi = 4$

$$\mathcal{D} = \{\underbrace{\{4,\dots,4\}}_{N^2 \text{ copies}}, \underbrace{\{2,\dots,2\}}_{2N^2 \text{ copies}}, \underbrace{\{4,\dots,4\}}_{N^2 \text{ copies}}\}.$$

Degree Sequences of Belyĭ Lattès map γ for $\deg\phi = 4$

Assume $N \equiv 1 \bmod 2$, then $0 \in B$, $1 \in W$, and $\infty \in F$. The degree sequence of γ is given by:

$$\mathcal{D} = \{\{1, \underbrace{4,\dots,4}_{\frac{N^2-1}{4} \text{ copies}}\}, \{1, \underbrace{2,\dots,2}_{\frac{N^2-1}{2} \text{ copies}}\}, \{1, \underbrace{4,\dots,4}_{\frac{N^2-1}{4} \text{ copies}}\}\}.$$

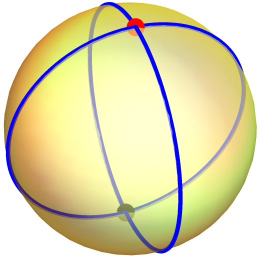
Assume $N \equiv 0 \bmod 2$, then $0, 1, \infty \in F$. The degree sequence of γ is given by:

$$\mathcal{D} = \{\{\underbrace{4,\dots,4}_{\frac{N^2}{4} \text{ copies}}\}, \{\underbrace{2,\dots,2}_{\frac{N^2}{2} \text{ copies}}\}, \{1, 1, 2, \underbrace{4,\dots,4}_{\frac{N^2-4}{4} \text{ copies}}\}\}.$$

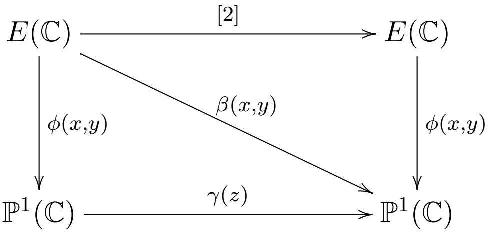
Monodromy from a Dessin D'Enfants

Assume that $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map of degree M. The monodromy group of ϕ is a subgroup of the symmetric group S_M as follows:

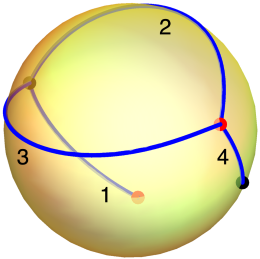
1. Label the edges of the Dessin d'Enfants of ϕ using 1 through M.
2. Write down the permutation σ_0 as the product of disjoint cycles found from reading the labels counterclockwise around each "black" vertex.
3. Write down the permutation σ_1 as the product of disjoint cycles found from reading the labels counterclockwise around each "red" vertex.
4. Generate the group $\mathrm{Mon}(\gamma) = \langle \sigma_0, \sigma_1 \rangle$ from these permutations.



- $E : y^2 = x^3 + 1$
- Belyĭ map $\phi(x,y) = \dfrac{1-y}{2}$ with degree 3
- multiplication-by-2 map $[2]$



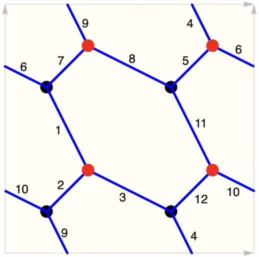
Dessin d'Enfants of Belyĭ Lattès Map $\gamma(z) = \frac{(z-1)(z+1)^3}{(2z-1)^3}$



$$\sigma_0 = (4)\,(1\ 3\ 2)$$
$$\sigma_1 = (1)\,(2\ 3\ 4)$$

$$\mathrm{Mon}\,\gamma = \langle \sigma_0, \sigma_1 \rangle \simeq A_4 \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \rtimes \frac{\mathbb{Z}}{3\mathbb{Z}}$$

Dessin d'Enfants of composition map $\beta(x,y) = \dfrac{(1+y)(3-y)^3}{16y^3}$



$$\sigma_0 = (1\ 7\ 6)\,(2\ 10\ 9)\,(3\ 4\ 12)\,(5\ 8\ 11)$$
$$\sigma_1 = (1\ 2\ 3)\,(4\ 5\ 6)\,(7\ 8\ 9)\,(10\ 11\ 12)$$

$$\operatorname{Mon}\beta = \langle \sigma_0, \sigma_1 \rangle \simeq A_4 \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \rtimes \frac{\mathbb{Z}}{3\mathbb{Z}}$$

How do we determine the group structure of $\operatorname{Mon}\gamma$ and $\operatorname{Mon}\beta$?

1. Calculate the Belyĭ Lattès map $\gamma(z)$ for a given $E(\mathbb{C})$ and $\phi(x, y)$.

   Do the same for composition map $\beta(x, y)$.

2. Generate the permutation cycles σ_0 and σ_1 from the map $\gamma(z)$.

   Apply the same method for $\beta(x, y)$.

3. Compute the group generated from σ_0 and σ_1 obtained in part (2) for $\gamma(z)$. Do the same for $\beta(x, y)$.

4. Compute the group identifier of the group found in part (3) using sage method. Find the group identifier associated to $\beta(x, y)$ and compare the two :)

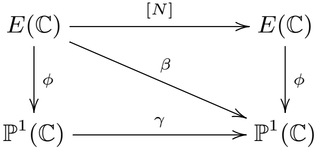
Setup and Motivating Question

Assume that $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function on an elliptic curve $E : y^2 = x^3 + Ax + B$, where $\deg(\phi) = M$, satisfying:

1. The following map is an isomorphism

$$\begin{array}{ccc} \mathbb{Z}/M\mathbb{Z} & \longrightarrow & \mathrm{Mon}(\phi) \\ m \mod M & \longmapsto & [\zeta_M^m](x,y) = \left(\zeta_M^{2m} x, \zeta_M^{3m} y\right). \end{array}$$

2. For all $N \in \mathbb{N}$, there exists some $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\gamma \circ \phi = \phi \circ [N]$, i.e. the following diagram commutes



Question

How are the monodromy groups of β and γ related?

Theorem

If the previous conditions are met, then the following are true:

1. *The composition* $\beta = \phi \circ [N] = \gamma \circ \phi$ *is a Belyĭ map, where* $\deg(\beta) = N^2 M$.
2. *The monodromy group of* β *is given by* $\mathrm{Mon}(\beta) = E[N] \rtimes \mathbb{Z}/M\mathbb{Z}$ *(here,* $E[N]$ *is the subgroup of order-N points).*
3. γ *is a Belyĭ Lattès map with* $\deg(\gamma) = N^2$*, and*

$$\mathrm{Mon}(\gamma) = E[N] \rtimes \left(\frac{d\mathbb{Z}}{M\mathbb{Z}}\right)$$

where

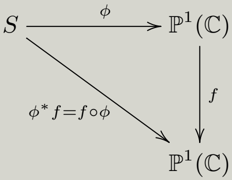
$$d = [\mathrm{Mon}(\beta) : \mathrm{Mon}(\gamma)] = \begin{cases} M & N = 1, \\ 2 & N = 2 \text{ and } M \text{ even}, \\ 1 & \text{otherwise.} \end{cases}$$

Function Fields and Degrees

Definition

Let S be a Riemann Surface, and let $\phi : S \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function.

- The **function field** $\mathcal{K}(S)$ is the set of all meromorphic maps $S \to \mathbb{P}^1(\mathbb{C})$.
- If $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is meromorphic, define the **pullback** $\phi^* : \mathcal{K}\left(\mathbb{P}^1(\mathbb{C})\right) \to \mathcal{K}(S)$ via $f(z) \mapsto f\left(\phi(x, y)\right)$.



- The **degree** of a map ϕ is the index $\deg \phi = \left[\ \mathcal{K}(S)\ :\ \phi^* \mathcal{K}\left(\mathbb{P}^1(\mathbb{C})\right)\ \right]$.

Definition

Let S be a Riemann Surface, and let $\phi : S \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function.

- The set of **Deck Transformations** of ϕ, which we call $\mathrm{Aut}(\phi)$, is defined as the automorphisms of S such that ϕ is preserved under composition:

$$\mathrm{Aut}(\phi) = \{\sigma \in \mathrm{Aut}\,(S) : \phi \circ \sigma = \phi\}\,.$$

Examples

$E(\mathbb{C})$	$\phi(x,y)$	Element of $\mathrm{Aut}(\phi)$	$\mathrm{Aut}(\phi)$
$y^2 = x^3 + 1$	$\frac{1-y}{2}$	$(x,y) \mapsto [\zeta_3^a](x,y) = \left(\zeta_3^{2a}x, \zeta_3^{3a}y\right)$	$\mathbb{Z}/3\mathbb{Z}$
$y^2 = x^3 - x$	x^2	$(x,y) \mapsto [\zeta_4^a](x,y) = \left(\zeta_4^{2a}x, \zeta_4^{3a}y\right)$	$\mathbb{Z}/4\mathbb{Z}$
$y^2 = x^3 + 1$	x^3	$(x,y) \mapsto [\zeta_6^a](x,y) = \left(\zeta_6^{2a}x, \zeta_6^{3a}y\right)$	$\mathbb{Z}/6\mathbb{Z}$

Lemma

Let $\deg(\phi) = M$, then $\mathrm{Aut}\,(\phi) \simeq \mathbb{Z}/M\mathbb{Z}$.

We prove that the map $\mathbb{Z}/M\mathbb{Z} \to \mathrm{Aut}(\phi)$ via a mod M $\mapsto$ $(P \mapsto [\zeta_M^a]P)$ is an isomorphism.

Lemma

$\mathrm{Aut}\,(\beta) \simeq E[N] \rtimes \mathbb{Z}/M\mathbb{Z}$, and $\mathrm{Mon}(\beta) = \mathrm{Aut}\,(\beta)$.

1. The following map is an isomorphism:

$$\begin{array}{ccc} E[N] \rtimes \mathbb{Z}/M\mathbb{Z} & \longrightarrow & \mathrm{Aut}\,\beta_N \\ (P_0\,,\ a \bmod \mathrm{M}) & \longmapsto & (P \mapsto [\zeta_M^a]P \oplus P_0)\,. \end{array}$$

2. A result from Zoladek states that $\mathrm{Mon}(\beta) = \mathrm{Aut}(\beta)$ iff $|\,\mathrm{Aut}(\beta)| = \deg(\beta)$.
3. Multiplicative property of degrees gives $\deg(\beta) = \deg(\phi) \cdot \deg([N]) = MN^2$.
4. Since $E[N] \simeq \frac{\mathbb{Z}}{N\mathbb{Z}} \times \frac{\mathbb{Z}}{N\mathbb{Z}}$, we have $|\,\mathrm{Aut}\,(\beta_N)\,| = MN^2 = \deg(\beta)$, thus $\mathrm{Mon}(\beta) = \mathrm{Aut}(\beta)$.

Definition

- Let L be a field, and let k be a subfield. Then we say L is a **field extension** of k, which we write as L/k.
- The dimension of L as a vector space over k, is the **degree of the extension** L/k, denoted $[L:k]$.

Proposition

- L/k is Galois if and only if $[L:k] = |\mathrm{Aut}(L/k)|$, where $\mathrm{Aut}(L/k)$ are the automorphisms of L which fix k. Then its **Galois Group** is $\mathrm{Gal}(L/k) = \mathrm{Aut}(L/k)$.
- In the following tower of extensions, $\mathrm{Aut}(L/k) = \mathrm{Aut}(\beta)$ and $\mathrm{Aut}(L/K) = \mathrm{Aut}(\phi)$.

$$\begin{array}{ccc}
L = \mathcal{K}\big(E(\mathbb{C})\big) & & \{\mathbb{1}\} \\
\Big|\, M & & \Big|\, M \\
K = \phi^*\mathcal{K}(\mathbb{P}^1(\mathbb{C})) & & H = \mathrm{Gal}(L/K) = \mathrm{Aut}(\phi) \simeq \mathbb{Z}/M\mathbb{Z} \\
\Big|\, N^2 & & \Big|\, N^2 \\
k = \beta^*\mathcal{K}(\mathbb{P}^1(\mathbb{C})) & & G = \mathrm{Gal}(L/k) = \mathrm{Aut}(\beta) \simeq E[N] \rtimes (\mathbb{Z}/M\mathbb{Z})
\end{array}$$

Theorem

Recall that $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$. Then K/k is Galois if and only if $H \lhd G$, in which case $\mathrm{Gal}(K/k) = G/H$.

$$\begin{array}{ccc} L = \mathcal{K}\big(E(\mathbb{C})\big) & \quad & \{\mathbb{1}\} \\ M \Big| & & M \Big| \\ K = \phi^* \mathcal{K}(\mathbb{P}^1(\mathbb{C})) & & H = \mathsf{Gal}(L/K) = \mathsf{Aut}(\phi) \\ N^2 \Big| & & N^2 \Big| \\ k = \beta^* \mathcal{K}(\mathbb{P}^1(\mathbb{C})) & & G = \mathsf{Gal}(L/k) = \mathsf{Aut}(\beta) \end{array}$$

Question

We want to find $\mathrm{Mon}(\gamma)$. Can't we just say $\mathrm{Mon}(\gamma) \simeq \mathrm{Gal}(K/k) \simeq G/H$?

Answer

Not necessarily! We don't know if $H \lhd G$ (i.e. if K/k is Galois). If H isn't normal in G, we can find the largest normal subgroup contained in H!

Proposition

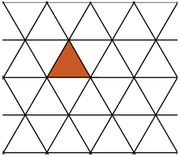
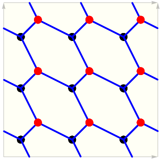
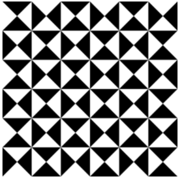
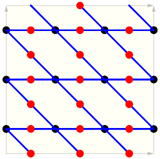
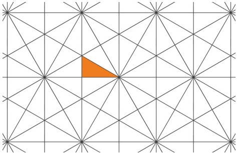
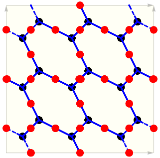
Recall that $G \simeq E[N] \rtimes {}^{\mathbb{Z}}/_{M\mathbb{Z}}$, and $H \simeq {}^{\mathbb{Z}}/_{M\mathbb{Z}}$. Then

$$\mathrm{Mon}(\gamma) = {}^{G}/_{\bar{N}}$$

$$\bar{N} = \cap_{\sigma \in G} \sigma H \sigma^{-1} = \begin{cases} \mathbb{Z}/M\mathbb{Z} & N = 1, \\ \mathbb{Z}/2\mathbb{Z} & N = 2 \text{ and M even}, \\ \bar{0} & \text{Otherwise.} \end{cases}$$

Corollary

$$\mathrm{Mon}(\gamma) = \begin{cases} \{\mathbb{1}\} & N = 1, \\ E[N] \rtimes \mathbb{Z}/(\frac{M}{2})\mathbb{Z} & N = 2 \text{ and } M \text{ even}, \\ \mathrm{Mon}(\beta) & \text{Otherwise.} \end{cases}$$

Triangles	Tilings of the Euclidean Plane	Tilings of the Torus
$60^\circ - 60^\circ - 60^\circ$		
$45^\circ - 45^\circ - 90^\circ$		
$30^\circ - 60^\circ - 90^\circ$		

Triangle Groups

1. Define the following angles of a triangle as π/l, π/n, and π/m.
2. Then we have the triangle group presentation

$$\Delta(l,m,n) = \langle a,b,c \,|\, a^2 = b^2 = c^2 = (ab)^l = (bc)^n = (ca)^m = 1\rangle.$$

Monodromy Groups

1. Assume that $G = E[N] \rtimes (\mathbb{Z}/M\mathbb{Z})$. With T_1 and $T_2 = [\zeta_M]T_1$ as generators of $E[N]$, denote the elements

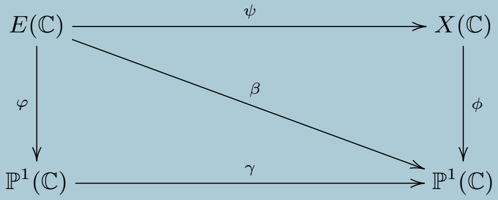
$$a = \big(T_1,\ 0 \bmod M\big), \quad b = \big(T_2,\ 0 \bmod M\big), \quad \text{and} \quad c = \big(O_E,\ 1 \bmod M\big).$$

Then we have the presentation

$$G = \left\langle a,\, b,\, c \;\middle|\; \begin{array}{c} a^N = b^N = c^M = 1, \\ a\,b = b\,a,\ c\,a\,c^{-1} = b,\ c\,b\,c^{-1} = a^{-1}\,b^{\varepsilon} \end{array} \right\rangle, \qquad \varepsilon = 2\,\cos\frac{2\pi}{M} = \begin{cases} -1 & \text{if } M = 3, \\ 0 & \text{if } M = 4, \text{ and} \\ +1 & \text{if } M = 6. \end{cases}$$

Future Work

What is the relationship between the monodromy groups of $\beta = \gamma \circ \varphi = \varphi \circ \psi$ and γ for an arbitrary isogeny $\psi : E(\mathbb{C}) \to X(\mathbb{C})$ between two elliptic curves E and X?



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